Synchronization of time-delayed systems

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In this paper we study synchronization in linearly coupled time-delayed systems. We first consider coupled nonidentical Ikeda systems with a square wave coupling rate. Using the theory of the time-delayed equation, we derive less restrictive synchronization conditions than those resulting from the Krasovskii-Lyapunov theory [Yang Kuang, *Delay Differential Equations* (Academic Press, New York, 1993)]. Then we consider a wide class of nonlinear nonidentical time-delayed systems. We also propose less restrictive synchronization conditions in an approximative sense, even if the coefficients in the linear time-delayed equation on the synchronization error are time dependent. Theoretical analysis and numerical simulations fully verify our main results.

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I. INTRODUCTION

Time-delayed systems are ubiquitous in nature, technology, and society because of finite signal transmission times, switching speeds, and memory effects [1]. It is well known that dissipative systems with a nonlinear time-delayed feedback or memory can produce chaotic dynamics [2,3], and the dimension of their chaotic attractors can be made arbitrarily large by increasing their delay time sufficiently [4]. Recently, the synchronization of the time-delayed systems has attracted much attention [5–12]. Complete synchronization [7], phase synchronization [8], anticipating synchronization [9], and generalized synchronization [10] are considered in the time-delayed coupled systems.

From recent works [1,6,9,11,12], the Krasovskii-Lyapunov theory [13] is useful for discussing the synchronization in the coupled time-delayed systems. According to this theory, some sufficient conditions are given to ensure synchronization in the coupled time-delayed systems [1,6,9,11]. However, these conditions are not valid for the general case where the linear time-delayed equation on the synchronization error are time varying; especially the coefficients in this equation are time dependent. This has been already shown in detail by Zhou *et al.* [12]. Therefore, it is necessary and important to propose the synchronization conditions for the above case.

In this paper we also consider the synchronization in the linearly coupled time-delayed systems. Here we study two cases of the coupled time-delayed systems. The first case is the coupled nonidentical Ikeda systems with a square wave coupling rate. Using the stability theory of the time-delayed equation, we can derive less restrictive synchronization conditions than those resulting from the Krasovskii-Lyapunov theory. The second case is a wide class of nonlinear nonidentical time-delayed systems. In an approximative sense, we also propose less restrictive synchronization conditions even if the coefficients in the linear time-delayed equation on the synchronization error are time dependent.

II. SYNCHRONIZATION IN THE COUPLED IKEDA SYSTEMS

First we analyze two nonidentical linearly coupled Ikeda systems

$$\frac{dx}{dt} = -\alpha x + m_1 \sin x(t-\tau_1), \tag{1}$$

$$\frac{dy}{dt} = -\alpha y + m_2 \sin y(t-\tau_2) + K(x-y), \tag{2}$$

where x is the phase lag of the electric field across the resonator, α is the relaxation coefficient, $m_{1,2}$ are the laser intensities injected into the systems, $\tau_{1,2}$ are the round-trip times of the light in the resonators or feedback delay times in the coupled systems, and K is the coupling rate between the drive and response systems [10]. The Ikeda model was introduced to describe the dynamics of an optical bistable resonator and is well known for delay-induced chaotic behavior [9,11].

Here we choose the coupling rate K as a square wave coupling, denoted by the following sequence;

$$\{(t_0,K_1),(t_1,K_2),(t_2,K_3),(t_3,K_4), \ldots\}, \tag{3}$$

where $t_j = t_0 + (j-1)\tau_s$ is the switching instant, the coupling rate $K_{2j-1} = k_1$ and $K_{2j} = k_2$ for all $j \geq 1$; k_1 and k_2 are different values. Within the interval $[t_{j-1}, t_j)$, the coupled Ikeda systems (1) and (2) become

$$\frac{dx}{dt} = -\alpha x + m_1 \sin x(t-\tau_1),$$

$$\frac{dy}{dt} = -\alpha y + m_2 \sin y(t-\tau_2) + K_j(x-y).$$

When the time delays $\tau_1 = \tau_2$, one necessary condition for complete synchronization in the coupled Ikeda systems (1) and (2) is $m_1 = m_2$ [10]. The dynamics of the error $\Delta = x - y$ is given by

$$\frac{d\Delta}{dt} = -(\alpha + K)\Delta - m_1 \cos x(t-\tau_1)\Delta(t-\tau_1). \tag{4}$$

It is obvious that $\Delta = 0$ is a solution of system (4).

To study the stability of system (4), one can design a Krasovskii-Lyapunov function as $V(t) = \frac{1}{2}\Delta^2(t) + \mu\int_{-\tau_1}^{0}\Delta^2(t+\theta)d\theta$, where $\mu > 0$ is an arbitrary positive parameter.

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The main purpose of the Krasovskii-Lyapunov theory is to find the condition for the negativeness of $\frac{dV}{dt}$ when the synchronization error Δ is not zero. Since $r(t)=\alpha+K$ and $s(t)=-m_1 \cos x(t-\tau_1)$ in system (4) are time varying, a sufficient condition for complete synchronization is $\alpha+K>|m_1|\geqslant|s(t)|$ for all time [1,6,9,11,12]. But this condition cannot be applied to the general case where $r(t)$ and $s(t)$ are time varying, especially $s(t)$ is not bounded for all time. Even for the square wave coupling rate, the above condition may not ensure complete synchronization. The main reason is that the parameter μ in the Krasovskii-Lyapunov function $V(t)$ is supposed to be time invariant. However, when $r(t)$ and $s(t)$ in the general case are time dependent, the parameter μ should be also time varying [12].

Here we propose a less restrictive synchronization condition using the stability theory of the time-delayed equation. Defining a positive-defined function $V(t)=\Delta^2(t)$, we get

$$\frac{dV}{dt} \leqslant -2(\alpha+K)V + 2|m_1||\Delta\Delta(t-\tau_1)| \leqslant -2\left(\alpha+K-\frac{|m_1|}{2}\right)V + |m_1|V(t-\tau_1). \quad (5)$$

Integrating the above inequality, we have

$$V(t) \leqslant V_0 + \int_{t_0}^{t}\left[-2\left(\alpha+K-\frac{|m_1|}{2}\right)V(s) + |m_1|V(s-\tau_1)\right]ds,$$

where V_0 is an initial condition of $V(t)$. From the comparison theorem of the delayed equation [14], the solution $V(t)$ satisfies

$$V(t) \leqslant \Gamma(t), \quad (6)$$

where $\Gamma(t)$ is the maximal solution of the following integral equation:

$$\Gamma(t) = V_0 + \int_{t_0}^{t}\left[-2\left(\alpha+K-\frac{|m_1|}{2}\right)\Gamma(s) + |m_1|\Gamma(s-\tau_1)\right]ds,$$

or equivalently,

$$\frac{d\Gamma}{dt} = -2(\alpha+K-|m_1|)\Gamma + \frac{|m_1|}{2}\Gamma(t-\tau_1), \quad (7)$$

with the same initial condition V_0. If we prove that $\lim_{t\to\infty}\Gamma(t)=0$, we get $\lim_{t\to\infty}V(t)=0$, which further results in the limit $\lim_{t\to\infty}\Delta(t)=0$. From Ref. [15], the stability of system (7) is equivalent to the stability of

$$\frac{d\Gamma_1}{dt} = \left[-2\left(\alpha+K-\frac{|m_1|}{2}\right) + \vartheta|m_1|\right]\Gamma_1, \quad (8)$$

where $\vartheta=\exp(j\theta), \theta\in[0,2\pi], j=\sqrt{-1}$. From the square wave coupling rate (3), if time t belongs to the interval $[t_{j-1},t_j)$, we get

$$\begin{aligned}\|\Gamma_1(t)\| &= \left\|\exp\left\{\int_{t_{j-1}}^{t}\left[-2\left(\alpha+K_j-\frac{|m_1|}{2}\right)+\vartheta|m_1|\right]ds\right\}\cdots \times\exp\left\{\int_{t_0}^{t_1}\left[-2\left(\alpha+K_1-\frac{|m_1|}{2}\right)+\vartheta|m_1|\right]ds\right\}V_0\right\| \\ &\leqslant \exp\left\{\int_{t_{j-1}}^{t}\left[-2\left(\alpha+K_j-\frac{|m_1|}{2}\right)+|m_1|\right]ds\right\}\cdots \times\exp\left\{\int_{t_0}^{t_1}\left[-2\left(\alpha+K_1-\frac{|m_1|}{2}\right)+|m_1|\right]ds\right\}V_0 \\ &= \exp\left\{2\int_{t_0}^{t}[-(\alpha+K-|m_1|)]ds\right\}V_0.\end{aligned}$$

Hence we obtain a sufficient condition for complete synchronization in the coupled Ikeda systems (1) and (2),

$$\int_{t_0}^{\infty}[-(\alpha+K-|m_1|)]dt = -\infty. \quad (9)$$

If this condition is satisfied, we get $\lim_{t\to\infty}\Gamma_1(t)=0$, which also means $\lim_{t\to\infty}\Gamma(t)=0$. However, this condition does not require that the derivative of a positive-defined function V is negative for all time when the synchronization error is not zero [1,6,9,11,12]. It aims to make $\lim_{t\to\infty}\Gamma(t)=0$, which also leads to $\lim_{t\to\infty}V(t)=0$. Further, condition (9) is less restrictive than the condition of $\alpha+K>|m_1|$ for all time [1,6,9,11]. For the square wave coupling (3), the condition for complete synchronization is $-(\alpha+k_1-|m_1|)\tau_s-(\alpha+k_2-|m_1|)\tau_s<0$ during one period. Complete synchronization in the coupled Ikeda systems (1) and (2) can also be ensured even if the coupling rate k_1 (or k_2) could not satisfy $\alpha+k_1>|m_1|$ (or $\alpha+k_2>|m_1|$).

When the time delays $\tau_1\neq\tau_2$, complete synchronization is not possible. But we can analyze the condition for generalized synchronization in the coupled Ikeda systems (1) and (2) based on the auxiliary approach [16]. Therefore, we should consider complete synchronization between the following Ikeda systems:

$$\frac{dy}{dt} = -\alpha y + m_2 \sin y(t-\tau_2) + K(x-y), \quad (10)$$

$$\frac{dz}{dt} = -\alpha z + m_2 \sin z(t-\tau_2) + K(x-z). \quad (11)$$

Applying the above approach for the case of complete synchronization, we get a sufficient condition for generalized synchronization in the coupled Ikeda systems (1) and (2) as follows:

$$\int_{t_0}^{\infty}[-(\alpha+K-|m_2|)]dt = -\infty, \quad (12)$$

where the coupling rate K is a square wave signal (3). This condition is also less restrictive than the condition of $\alpha+K>|m_2|$ for all time [1,6,9,11].

III. SYNCHRONIZATION IN THE NONLINEAR TIME-DELAYED SYSTEMS

Now we generalize the above approach to a wide class of nonlinear nonidentical time-delayed systems. Consider a general form of one-way coupled scalar time-delayed systems,

$$\frac{dx}{dt} = F(p,x,x_{\tau_1}), \tag{13}$$

$$\frac{dy}{dt} = F(q,y,y_{\tau_2}) + K(x-y), \tag{14}$$

where K is the coupling rate, and p and q are the parameters for the drive system and the response system, respectively.

When the delays $\tau_1 = \tau_2$ and the parameters $p=q$, we consider complete synchronization in the coupled systems (13) and (14). A small deviation $\Delta = x - y$ is governed by the linearized time-delayed equation

$$\frac{d\Delta}{dt} = -r(t)\Delta + s(t)\Delta(t-\tau_1), \tag{15}$$

where $r(t)=(K-\partial_x)F(p,x,x_{\tau_1})$ and $s(t)=\partial_{x_{\tau_1}}F(p,x,x_{\tau_1})$.

Similarly, one can also design a Krasovskii-Lyapunov function as $V(t)=\frac{1}{2}\Delta^2(t)+\mu\int_{-\tau_1}^{0}\Delta^2(t+\theta)d\theta$ to study the stability of system (15). Further, a sufficient condition for complete synchronization is $r(t) > |s(t)|$ for all time [1,6,9,11,12]. However, this condition cannot be applied to the general case where $r(t)$ and $s(t)$ are time varying [12]. Here we give one sufficient synchronization condition in an approximative sense.

We first segment the time interval $[t_0,\infty)$ into $[t_0,\infty) = \cup_{j\geqslant 1}[t_{j-1},t_j)$, where $\tau_0 > 0$ is sufficiently small, $t_j = t_0 + j\tau_0$, and τ_1 is a multiple of τ_0. If τ_0 is sufficiently small, the solution $x(t)$ of the drive system (13) can be approximated by $x(t_j)$ within the interval $[t_{j-1},t_j)$, which further results in the approximation of $r(t),s(t)$ by $r(t_j),s(t_j)$, respectively. Therefore, within the interval $[t_{j-1},t_j)$, system (15) can be approximated by

$$\frac{d\Delta}{dt} = -r'(t)\Delta + s'(t)\Delta(t-\tau_1), \tag{16}$$

where $r'(t)=r(t_j)$ and $s'(t)=s(t_j)$. In fact, the above approximation of system (15) can be regarded as the following sequence:

$$\{(t_0,\{s'(t_1),r'(t_1)\}),(t_1,\{s'(t_2),r'(t_2)\}),$$
$$(t_2,\{s'(t_3),r'(t_3)\}),\ \ldots\}. \tag{17}$$

Applying the approach for the case of the coupled Ikeda systems, we can derive the stability condition for the approximation system (16) as follows:

$$\int_{t_0}^{\infty}\{-[r'(t)-|s'(t)|]\}dt = -\infty, \tag{18}$$

where $r'(t),s'(t)$ satisfy the sequence (17). As τ_0 tends to zero, the stability of system (16) becomes the stability of system (15). Therefore, if τ_0 is sufficiently small, condition (18) can be approximately regarded as a condition for complete synchronization of the coupled systems (13) and (14).

When the delays $\tau_1 \neq \tau_2$, we can also study generalized synchronization in the coupled systems (13) and (14). From the auxiliary approach [16], we should consider complete synchronization between the following systems:

$$\frac{dy}{dt} = F(q,y,y_{\tau_2}) + K(x-y), \tag{19}$$

$$\frac{dz}{dt} = F(q,z,z_{\tau_2}) + K(x-z). \tag{20}$$

The small deviation $\Delta' = y - z$ is governed by

$$\frac{d\Delta'}{dt} = -r(t)\Delta' + s(t)\Delta'(t-\tau_2), \tag{21}$$

where $r(t)=(K-\partial_x)F(q,x,x_{\tau_2})$ and $s(t)=\partial_{x_{\tau_2}}F(q,x,x_{\tau_2})$. Similarly, we segment the time interval $[t_0,\infty)$ into $[t_0,\infty) = \cup_{j\geqslant 1}[t_{j-1},t_j)$, where τ_0 is sufficiently small, and τ_2 is the multiple of τ_0. Therefore, system (21) can be approximated sufficiently by

$$\frac{d\Delta'}{dt} = -r'(t)\Delta + s'(t)\Delta'(t-\tau_2), \tag{22}$$

in which $r'(t)=r(t_j)$ and $s'(t)=s(t_j)$ within the interval $[t_{j-1},t_j)$. We also derive the stability condition for system (22)

$$\int_{t_0}^{\infty}\{-[r'(t)-|s'(t)|]\}dt = -\infty, \tag{23}$$

where $r'(t),s'(t)$ satisfy the sequence (17). If τ_0 is sufficiently small, condition (23) can be also approximately regarded as a condition for generalized synchronization of the coupled systems (13) and (14).

From the above analysis, if τ_0 approaches zero, $x(t_j),r'(t),s'(t)$ approach $x(t),r(t),s(t)$, respectively, and system (16) [or system (22)] can also approximate system (15) [or system (21)]. If τ_0 tends to zero, conditions (18) and (23) become

$$\int_{t_0}^{\infty}\{-[r(t)-|s(t)|]\}dt = -\infty, \tag{24}$$

where $r(t)=(K-\partial_x)F(p,x,x_{\tau_1})$ and $s(t)=\partial_{x_{\tau_1}}F(p,x,x_{\tau_1})$ for complete synchronization, and $r(t)=(K-\partial_x)F(q,x,x_{\tau_2})$ and $s(t)=\partial_{x_{\tau_2}}F(q,x,x_{\tau_2})$ for generalized synchronization. Hence condition (24) can be considered as a sufficient synchronization condition in an approximative sense.

IV. NUMERICAL SIMULATIONS

We choose two typical examples to confirm our main results. One is the coupled nonidentical Ikeda systems; the other is the coupled nonidentical Mackey-Glass systems [2].

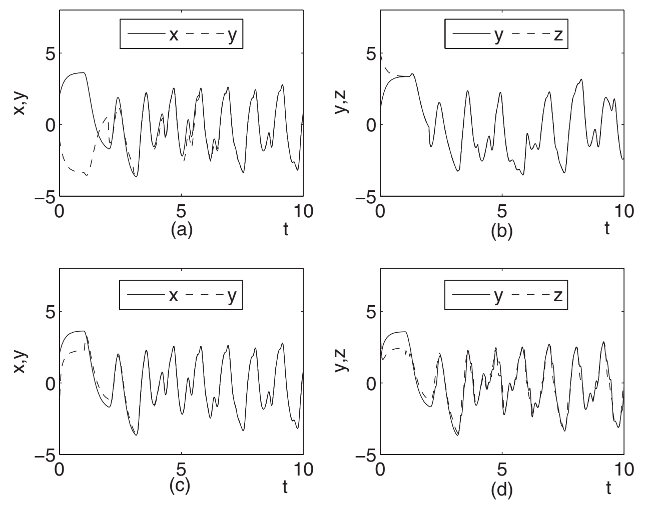

FIG. 1. Complete synchronization and generalized synchronization in the coupled Ikeda systems (1) and (2). (a) Complete synchronization for $\tau_1=\tau_2=1$, $\alpha=5$, $m_1=m_2=20$, and the square wave coupling rate {(0, 0),(2, 40),(4, 0),(6, 40),…}. (b) Generalized synchronization for $\tau_1=1$, $\tau_2=1.2$ [or complete synchronization in systems (10) and (11)]. All the other parameters have the same values as those in Fig. 1(a). (c) Complete synchronization in the coupled Ikeda systems (1) and (2) where the coupling rate $K(t)=-\alpha+2m_1|\cos x(t-\tau_1)|$. All the other parameters have the same values as those in Fig. 1(a). (d) Generalized synchronization in the coupled Ikeda systems (1) and (2) where the coupling rate $K(t)=-\alpha+2m_2|\cos x(t-\tau_2)|$ [or complete synchronization in the systems (10) and (11)]. All the other parameters have the same values as those in Fig. 1(b).

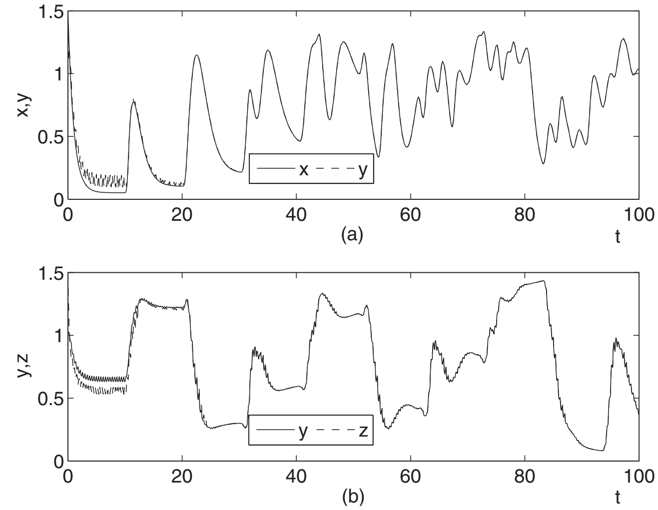

FIG. 2. Complete synchronization and generalized synchronization in the coupled Mackey-Glass systems (25) and (26). (a) Complete synchronization for $a_1=a_2=2$, $b_1=b_2=10$, $c_1=c_2=1$, $\tau_1=\tau_2=10$. (b) Generalized synchronization for $a_1=a_2=2$, $b_1=b_2=10$, $c_1=c_2=1$, $\tau_1=10$, $\tau_2=12$ [or complete synchronization in the systems $\frac{dy}{dt}=-c_2y+a_2y_{\tau_2}/(1+y_{\tau_2}^{b_2})+K(x-y)$ and $\frac{dz}{dt}=-c_2z+a_2z_{\tau_2}/(1+z_{\tau_2}^{b_2})+K(x-z)$].

Note that all the results are simulated using the DDE23 program [17] in MATLAB 7.

Example 1. The coupled nonidentical Ikeda systems (1) and (2). We first choose a square wave coupling rate K as the sequence {(0, 0),(2, 40),(4, 0),(6, 40),…}. Obviously, within the intervals [0, 2),[4, 6),…, there exists no coupling (namely, the coupling rate $K=0$), which means that $\alpha>|m_1|$ for complete synchronization and $\alpha>|m_2|$ for generalized synchronization are not satisfied within these intervals. Hence conditions resulting from the Krasovskii-Lyapunov theory cannot ensure the synchronization. However, after a simple computation, condition (9) for complete synchronization and condition (12) for generalized synchronization are satisfied. Figure 1(a) shows complete synchronization for $\tau_1=\tau_2=1$, $\alpha=5$, and $m_1=m_2=20$. Figure 1(b) shows generalized synchronization in the coupled Ikeda systems (1) and (2) for $\tau_1=1$, $\tau_2=1.2$.

For the general case of the coupling rate $K(t)$, we still derive the approximated synchronization condition (24) where $r(t)=\alpha+K(t)$ and $s(t)=-m_1\cos x(t-\tau_1)$ for complete synchronization, and $r(t)=\alpha+K(t)$ and $s(t)=-m_2\cos x(t-\tau_2)$ for generalized synchronization. For complete synchronization, we choose $K(t)=-\alpha+2m_1|\cos x(t-\tau_1)|$. Hence $r(t)=2m_1|\cos x(t-\tau_1)|\geqslant|s(t)|=m_1|\cos x(t-\tau_1)|$. From Refs. [1,6,9,11,12], complete synchronization cannot be ensured by the condition of $r(t)>m_1\geqslant|s(t)|$ (or $r(t)>|s(t)|$) for all time. However, we get $\int_{t_0}^{\infty}\{-[r(t)-|s(t)|]\}dt=\int_{t_0}^{\infty}[-m_1|\cos x(t-\tau_1)|]dt=-\infty$ due to the chaotic nature of state x. Figure 1(c) shows the simulation result for complete synchronization. For generalized synchronization, we choose $K(t)=-\alpha+2m_2|\cos x(t-\tau_2)|$. Hence $r(t)=2m_2|\cos x(t-\tau_2)|\geqslant|s(t)|=m_2|\cos x(t-\tau_2)|$. Since the condition of $r(t)>m_2\geqslant|s(t)|$ [or $r(t)>|s(t)|$] for all time is not satisfied, generalized synchronization cannot be ensured by synchronization conditions [1,6,9,11,12]. However, we still have $\int_{t_0}^{\infty}\{-[r(t)-|s(t)|]\}dt=\int_{t_0}^{\infty}[-m_2|\cos x(t-\tau_2)|]dt=-\infty$. Figure 1(d) shows the simulation result for generalized synchronization.

Example 2. The coupled Mackey-Glass systems. The coupled nonidentical Mackey-Glass systems [2,6] are

$$\frac{dx}{dt}=-c_1x+a_1x_{\tau_1}/(1+x_{\tau_1}^{b_1}), \qquad (25)$$

$$\frac{dy}{dt}=-c_2y+a_2y_{\tau_2}/(1+y_{\tau_2}^{b_2})+K(x-y). \qquad (26)$$

Initially the Mackey-Glass system has been introduced as a model of blood generation for patients with leukemia. Later this model became popular in chaos theory as a model for producing high dimensional chaos to test various methods of chaotic time series analysis, controlling chaos, etc.

We consider the synchronization in the coupled systems (25) and (26), where the square wave coupling rate K is denoted by the sequence $\{(0,-6),(0.2,30),(0.4,-6),(0.6,30),\dots\}$. In this case we set $a_1=a_2=2$, $b_1=b_2=10$, $c_1=c_2=1$, $\tau_1=\tau_2=10$ for complete synchronization, and $\tau_1=10$, $\tau_2=12$ for generalized synchronization. Here $r(t)=K+c_1$ is a special kind of time-varying coupling rates, $s(t)=\frac{\partial}{\partial x_{\tau_1}}[a_1x_{\tau_1}/(1+x_{\tau_1}^{b_1})]$ for complete synchronization and $s(t)$

$=\frac{\partial}{\partial x_{\tau_2}}[a_2 x_{\tau_2}/(1+x_{\tau_2}^{b_2})]$ for generalized synchronization are also time varying. After a simple computation, we get $|\frac{\partial}{\partial x_\tau}[a_1 x_\tau/(1+x_\tau^{b_1})]| \leqslant a_1+\frac{a_1 b_1}{2}=12$ for any $\tau>0$. Since the coupling rate K within the intervals [0, 0.2),[0.4, 0.6),… is negative, the condition of $r(t)>m_2 \geqslant |s(t)|$ (or $r(t)>|s(t)|$) for all time is not satisfied [1,6,9,11,12]. Therefore we should verify the satisfaction of condition (24) for complete synchronization and generalized synchronization. It is easy to verify that $\int_{t_0}^{\infty}\{-[K+c_1-|\frac{\partial}{\partial x_{\tau_1}}f(x_{\tau_1})|]\}dt=-\infty$ and $\int_{t_0}^{\infty}\{-[K+c_1-|\frac{\partial}{\partial x_{\tau_2}}f(x_{\tau_2})|]\}dt=-\infty$. Hence complete synchronization (24) for $\tau_1=\tau_2$ and generalized synchronization (24) for $\tau_1 \neq \tau_2$ in the coupled Mackey-Glass systems (25) and (26) are ensured.

The simulation results are plotted in Fig. 2. Figure 2(a) shows complete synchronization in the coupled nonidentical Mackey-Glass systems (25) and (26) when $a_1=a_2=2$, $b_1=b_2=10$, $c_1=c_2=1$, and $\tau_1=\tau_2=10$. Figure 2(b) shows generalized synchronization in the coupled nonidentical Mackey-Glass systems (25) and (26) for $a_1=a_2=2$, $b_1=b_2=10$, $c_1=c_2=1$, $\tau_1=10$, and $\tau_2=12$.

V. CONCLUSION

We have studied synchronization in linearly coupled time-delayed systems. We first consider the coupled nonidentical Ikeda systems with a square wave coupling rate. We derive less restrictive synchronization conditions than those resulting from the Krasovskii-Lyapunov theory. Then we generalize the above approach to a wide class of nonlinear nonidentical time-delayed systems. Even if the coefficients in the linear time-delayed equation on the synchronization error are time dependent, we also propose less restrictive synchronization conditions in an approximative sense. Numerical simulations of the coupled Ikeda systems and the coupled Mackey-Glass systems fully verify our main results.

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